

Random field and random anisotropy $O(N)$ spin systems with a free surface

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We study the surface scaling behavior of a semi-infinite d -dimensional $O(N)$ spin system in the presence of quenched random field and random anisotropy disorders. It is known that above the lower critical dimension $d_{lc} = 4$ the infinite models undergo a paramagnetic-ferromagnetic transition for $N > N_c$ ($N_c = 2.835$ for random field and $N_c = 9.441$ for random anisotropy). For $N < N_c$ and $d < d_{lc}$ there exists a quasi-long-range ordered phase with zero order parameter and a power-law decay of spin correlations. Using functional renormalization group we derive the surface scaling laws which describe the ordinary surface transition for $d > d_{lc}$ and the long-range behavior of spin correlations near the surface in the quasi-long-range ordered phase for $d < d_{lc}$. The corresponding surface exponents are calculated to one-loop order. The obtained results can be applied to the surface scaling of periodic elastic systems in disordered media and amorphous magnets.

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I. INTRODUCTION

The phase diagram and critical properties of spin systems with quenched disorder attracted considerable interest for decades. One usually distinguishes two types of quenched disorder: (i) random-temperature like disorder corresponding to randomness coupled to the local energy density as, for example, in diluted ferromagnets [1]; (ii) random field like disorder corresponding to the case when the order parameter couples to a random symmetry breaking field [2]. The influence of random-temperature disorder is rather well understood. There exist several powerful methods to study the phase behavior and criticality such as perturbative renormalization group (RG). The effect of the random field disorder being more profound is much less studied. The prominent example is the critical behavior of the random field Ising model (RFIM) which complete understanding is still lacking despite significant numerical, analytical and experimental efforts [3]. It has been found that the perturbative calculations including standard RG methods lead to incorrect results, in particular, to the so-called dimensional reduction (DR). Analysis of the Feynman diagrams giving the leading singularities [4] or using supersymmetry [5] predicts that the critical behavior of the RFIM in d dimension is the same as that of the pure system in $d - 2$ dimensions. Consequently, the lower critical dimension of the RFIM below which there is no true long-range order is expected to be $d_{lc}^{DR} = 3$. However, the simple Imry-Ma arguments show that the lower critical dimension of the RFIM is in fact $d_{lc} = 2$ [2]. The deviation from the DR prediction is also confirmed by the high-temperature expansion [6] and real space RG [7]. The failure of DR can be explained by complicated energy landscape which renders the perturbation theory spoiled to all orders by unphysical averaging over multiple minima and maxima. The latter can be formulated in terms of supersymmetry or replica symmetry breaking [8, 9]. To overcome this obstacle one needs a non-perturbative method or correct resumming the perturbation theory.

The considerable progress has been achieved last years in studying the $O(N)$ models in which disorder couples to the N -component order parameter either linearly as in the random field (RF) case or bilinearly as in a random anisotropy

(RA) system. These models are relevant for diverse physical systems including amorphous magnets [10], diluted antiferromagnets in a uniform external magnetic field [11], liquid crystals in porous media [12, 13], nematic elastomers [14], critical fluids in aerogels [15–17], vortices in type II superconductors [18], and stochastic inflation in cosmology [19]. Similar to the RFIM these models suffer of DR [4, 20]. It was shown that the expansion around the lower critical dimension of the the RF $O(N)$ model $d_{lc} = 4$ generates an infinite number of relevant operators which can be parameterized by a single function [21]. However, the RG flow of this function has no analytic fixed point (FP). Only almost two decades later, being inspired by the progress in disordered elastic systems [22–25], it was realized that the scaling properties of the RF and RA systems are encoded in a nonanalytic FP [26]. The non-analytic FPs control the paramagnetic-ferromagnetic phase transitions in the RF and RA $O(N)$ model and allow one to compute the critical exponents within $\varepsilon = d - 4$ expansion [27]. The obtained exponents are different from the DR prediction. The FRG calculations have been extended to two-loop order [28, 29] and the effect of long-range disorder correlations has been studied [17, 30]. Using developed in Ref. [31] truncated exact FRG it was argued that spontaneous breaking of the supersymmetry which leads to a breakdown of DR, occurs only below a critical dimension $d_{DR} \approx 5.1$ [32].

A more peculiar issue concerns the phase diagram of the RF and RA models below d_{lc} . It is known that for the RF model and models with isotropic distributions of random anisotropies true long-range order is forbidden below $d_{lc} = 4$ (for anisotropic distributions, long-range order can occur even below d_{lc} [33]). Nevertheless, quasi-long-range order (QLRO) with zero order parameter and an infinite correlation length can persist even for $d_{lc}^*(N) < d < d_{lc}$, where $d_{lc}^*(N)$ is the lower critical dimension for the paramagnetic-QLRO transition. For example, the Gaussian variational approximation predicts that the vortex lattice in disordered type-II superconductors can form the so-called Bragg glass exhibiting slow logarithmic growth of displacements [34]. This system can be mapped onto the three dimensional RF $O(2)$ model, in which the Bragg glass corresponds to the QLRO phase. Indeed, for $N < N_c$ and $d < d_{lc}$, the FRG equations have attractive FPs

which describe the QLRO phases of RF and RA models [26]. Despite that the question of the lower critical dimension of the paramagnetic-QLRO transition is still controversial. In order to study the transition between the QLRO phase and the disordered phase using FRG, one has to go beyond the one-loop approximation. The truncated exact FRG [31] and the two-loop FRG [29] performed using a double expansion in $\sqrt{|\varepsilon|}$ and $N - N_c$ provide an additional singly unstable FP which controls the transition. Both methods give qualitatively similar pictures of the FRG flows: the critical and attractive FPs merge in some dimension $d_{lc}^*(N) < d_{lc}$ which is considered as the lower critical dimension of the paramagnetic-QLRO transition. For the RF $O(2)$ model, both methods give approximately the same estimation $d_{lc}^* \approx 3.8(1)$, and thus, suggest that there is no Bragg glass phase in $d = 3$. However, one has to take caution when extrapolating results obtained for small $\sqrt{|\varepsilon|}$ and $N - N_c$. Moreover, in contrast to the model of Refs. [26] and [29] which belongs to the so-called “hard-spin” models, the system studied in Ref. [31] corresponds to “soft spins”. They can belong to different universality classes since the soft spin model allows for topological defects which destroy the QLRO.

The real systems, usually, are finite and have boundaries which effect is twofold: (i) the free energy of the system in addition to the bulk contribution proportional to the volume acquires a new term proportional to the area of the surface; (ii) the presence of boundaries breaks the translational invariance. In general, this can modify the behavior in the boundary region extended in the bulk only over distances of the order of the bulk correlation length. However, at the bulk critical point or in the QLRO phase, the bulk correlation length is infinite so that one can expect that the effect of boundaries to be more pronounced. Indeed, the presence of the boundaries introduces a whole set of critical exponents describing the scaling behavior at and close to the boundary at criticality [35]. Several different classes of the surface transitions are known depending upon boundary conditions [36]. The *ordinary transition* corresponds to the case when the surface magnetization is suppressed due to reduced number of close neighbors near the boundary so that the surface ordering is completely driven by the bulk magnetization. If for some reason the coupling between spins on the surface is sufficiently enhanced with respect to the bulk coupling or there is an external surface magnetic field, the surface may order before the bulk does. The latter is called the *surface transition*. Then the system can undergo the so-called *extraordinary transition* in the presence of ordered surface. The two lines of the extraordinary transition and the surface transition meet at the multicritical point which is called the *special transition*. The last three transition can take place only if the dimension of the surface $d - 1$ is above the lower critical dimension for the transition. These transitions have been studied for various systems with discrete and continuous symmetries using different methods, such as RG and numerical simulations (for review see [35, 37, 38]).

The effect of weak random temperature like disorder on the surface criticality was studied using RG methods in Refs. [39, 40]. However, not so much is known about the surface criticality in systems with RF disorder. The phase di-

agram of the 3D semi-infinite RFIM as a function of the ratio of bulk and surface interactions and the ratio of bulk and surface fields has been studied using a mean field approximation in Ref. [41]. The surface criticality of the RFIM has been studied numerically in Ref. [42]. It was also shown that the RF disorder on the surface of a 3D spin system with continuous symmetry destroys the long-range order in the bulk, and, instead, a QLRO emerges [43]. In this work we address the question how do the RF and RA disorder in the bulk effect the behavior of spin systems with continuous symmetry in vicinity of free surfaces. In particular, we consider the ordinary surface transition of the RF and RA $O(N)$ models for $d > 4$ and the spin correlations in the QLRO phase near a free surface for $d < 4$.

The paper is organized as follows. Section II introduces the model. In Sec. III we renormalize the theory and derive the scaling laws. In Sec. IV we calculate the surface critical exponents to one-loop order. Section V summarizes the obtained results.

II. MODEL AND SCALING LAWS

We consider a d -dimensional semi-infinite $O(N)$ spin system which configuration is given by the N -component classical vector field $\mathbf{s}(\mathbf{r})$ satisfying the fixed-length constraint $|\mathbf{s}(\mathbf{r})|^2 = 1$. The position vector $\mathbf{r} = (\mathbf{x}, z)$ has a $(d - 1)$ -dimensional component \mathbf{x} parallel to the surface and a one-dimensional component $z \geq 0$ which is perpendicular to the surface $z = 0$. It is convenient to introduce short notations for the volume integral over half space $\int_V := \int_0^\infty dz \int d^{d-1}x$ and for the surface integral $\int_S := \int d^{d-1}x$. The large-scale behavior of the disordered spin system can be described by the effective Hamiltonian

$$\mathcal{H}[\mathbf{s}] = \mathcal{H}_0[\mathbf{s}] + \mathcal{H}_{\text{surf}}[\mathbf{s}] + \mathcal{H}_{\text{dis}}[\mathbf{s}], \quad (1)$$

consisting of the sum of three terms which result from the semi-infinite bulk, surface and disorder in the bulk. The contribution from the the semi-infinite bulk can be expressed in the form of the well-known $O(N)$ nonlinear sigma model:

$$\mathcal{H}_0[\mathbf{s}] = \int_V \left[\frac{1}{2} (\nabla \mathbf{s}(\mathbf{r}))^2 - \mathbf{h} \cdot \mathbf{s}(\mathbf{r}) \right], \quad (2)$$

where \mathbf{h} is the magnetic field in the bulk. The surface contribution to Hamiltonian can be written in its simplest form as [44]:

$$\mathcal{H}_{\text{surf}}[\mathbf{s}] = - \int_S \mathbf{h}_1 \cdot \mathbf{s}(\mathbf{x}), \quad (3)$$

where for simplicity we assume that the surface magnetic field \mathbf{h}_1 has the same direction as the bulk field \mathbf{h} . We consider a quite general type of bulk disorder such that its potential can be expanded in spin variables as follows

$$\mathcal{H}_{\text{dis}}[\mathbf{s}] = - \int_V \sum_{\mu=1}^{\infty} \sum_{i_1 \dots i_\mu} h_{i_1 \dots i_\mu}^{(\mu)}(\mathbf{r}) s_{i_1}(\mathbf{r}) \dots s_{i_\mu}(\mathbf{r}). \quad (4)$$

The coefficients $h_{i_1 \dots i_\mu}^{(\mu)}(\mathbf{r})$ are assumed to be Gaussian random variables with zero mean and variances given by

$$\overline{h_{i_1 \dots i_\mu}^{(\mu)}(\mathbf{r}) h_{i_1' \dots i_\mu'}^{(\mu')}(\mathbf{r}')} = \delta^{\mu\mu'} \delta_{i_1 j_1} \dots \delta_{i_\mu j_\mu} r_\mu \delta(\mathbf{r} - \mathbf{r}'). \quad (5)$$

The first two coefficients have simple physical interpretation: $h_i^{(1)}$ is a random magnetic field and $h_{ij}^{(2)}$ is a second-rank random anisotropy. The higher order coefficients $h^{(\mu)}$ are higher order random anisotropies. As was shown in Ref. 21, even if the system has only finite number of nonzero bare $h^{(\mu)}$, the RG transformations will generate an infinite set of higher-order anisotropies. However, the RG flow preserves the symmetry with respect to rotation $\mathbf{s} \rightarrow -\mathbf{s}$. For instance, starting from the bare model with only a second-rank anisotropy only even-rank anisotropies will be generated by the RG flow. We will reserve the notation RA for the systems which possess this symmetry and the notation RF for the systems which do not.

We employ the replica trick to average over disorder. Introducing n replicas of the original system and averaging their joint partition function over disorder we obtain the replicated Hamiltonian

$$\mathcal{H}_n = \int_V \left\{ \sum_{a=1}^n \left[\frac{1}{2} (\nabla \mathbf{s}_a(\mathbf{r}))^2 - \mathbf{h} \cdot \mathbf{s}_a(\mathbf{r}) \right] - \frac{1}{2T} \sum_{a,b=1}^n \mathcal{R}(\mathbf{s}_a(\mathbf{r}) \cdot \mathbf{s}_b(\mathbf{r})) \right\} - \sum_{a=1}^n \int_S \mathbf{h}_1 \cdot \mathbf{s}_a(\mathbf{x}), \quad (6)$$

where we have defined the function $\mathcal{R}(z) = \sum_\mu r_\mu z^\mu$. The properties of the original disordered system (1) can be extracted in the limit $n \rightarrow 0$. According to the above definition of the RF and RA models, the function $\mathcal{R}(z)$ is arbitrary in the case of the RF model and even for the RA systems.

Power counting shows that $d_{lc} = 4$ is the lower critical dimension of the model (6) [20]. Above the lower critical dimension the RF and RA systems undergo a paramagnetic-ferromagnetic transition. The scaling behavior at criticality is controlled by a zero temperature fixed point (FP) similar to the RFIM [45], reflecting the fact that disorder dominates over the thermal fluctuations. However, the temperature is dangerously irrelevant. For instance, this results in violation of the usual hyperscaling relation and appearance of an additional universal exponent θ that modifies the hyperscaling relation to [3]:

$$\nu(d - \theta) = 2 - \alpha, \quad (7)$$

where ν and α are the correlation length and the specific heat exponents. One also expects a dramatic slowing down as the transition is approached with the characteristic relaxation time $\ln \tau \sim t_1^{-\nu\theta}$, where $t_1 = |T - T_c|/T_c$ is the reduced temperature [46]. The magnetization in the bulk and on the surface vanish at the transition according to

$$\sigma(t_1) \sim t_1^\beta, \quad \sigma_1(t_1) \sim t_1^{\beta_1}, \quad (8)$$

where we have introduced the bulk and the surface magnetization exponents. At the critical point $t_1 = 0$ a small magnetic

field in the bulk \mathbf{h} induces the magnetization in the bulk and also on the surface according to

$$\sigma(h) \sim h^{1/\delta}, \quad \sigma_1(h) \sim h^{1/\delta_1}, \quad (9)$$

where we define the exponents δ and δ_1 . The surface magnetic field \mathbf{h}_1 leads to the surface magnetization

$$\sigma_1(h_1) \sim h_1^{1/\delta_{11}}. \quad (10)$$

Below the lower critical dimension d_{lc} a QLRO phase with zero magnetization can emerge. At criticality or in the QLRO phase, the correlation functions of the order parameter exhibit scaling behavior. Due to dangerous irrelevance of the temperature the connected and disconnected correlation functions scale with different exponents. We define the connected and disconnected correlation functions of the two local operators A and B as

$$[A(\mathbf{r}) \cdot B(\mathbf{r}')]_{\text{con}} := \overline{\langle A(\mathbf{r}) \cdot B(\mathbf{r}') \rangle} - \langle A(\mathbf{r}) \rangle \cdot \langle B(\mathbf{r}') \rangle, \\ [A(\mathbf{r}) \cdot B(\mathbf{r}')]_{\text{dis}} := \overline{\langle A(\mathbf{r}) \rangle \cdot \langle B(\mathbf{r}') \rangle} - \langle A(\mathbf{r}) \rangle \cdot \langle B(\mathbf{r}') \rangle.$$

Here the angular brackets denote the thermal averaging and the bar stands for the disorder averaging. For instance, the connected and disconnected correlation functions of spins in the bulk scale independently as

$$[\mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{r}')]_{\text{con}} \sim \frac{1}{|\mathbf{r} - \mathbf{r}'|^{d-2+\eta}}, \quad (11)$$

$$[\mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{r}')]_{\text{dis}} \sim \frac{1}{|\mathbf{r} - \mathbf{r}'|^{d-4+\bar{\eta}}}. \quad (12)$$

Following the general scaling picture of the surface critical phenomena we introduce the surface exponents η_\perp and $\bar{\eta}_\perp$ which replace the bulk exponents η and $\bar{\eta}$ in Eqs. (11) and (12) when one of the points \mathbf{r} or \mathbf{r}' belongs to the surface:

$$[\mathbf{s}(\mathbf{x}, z) \cdot \mathbf{s}(\mathbf{x}', 0)]_{\text{con}} \sim \frac{1}{((\mathbf{x} - \mathbf{x}')^2 + z^2)^{(d-2+\eta_\perp)/2}}, \quad (13)$$

$$[\mathbf{s}(\mathbf{x}, z) \cdot \mathbf{s}(\mathbf{x}', 0)]_{\text{dis}} \sim \frac{1}{((\mathbf{x} - \mathbf{x}')^2 + z^2)^{(d-4+\bar{\eta}_\perp)/2}}. \quad (14)$$

We also define the surface exponents η_\parallel and $\bar{\eta}_\parallel$ which describe the connected and disconnected correlation function when the both points lie on the surface:

$$[\mathbf{s}(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}')]_{\text{con}} \sim \frac{1}{|\mathbf{x} - \mathbf{x}'|^{d-2+\eta_\parallel}}, \quad (15)$$

$$[\mathbf{s}(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}')]_{\text{dis}} \sim \frac{1}{|\mathbf{x} - \mathbf{x}'|^{d-4+\bar{\eta}_\parallel}}. \quad (16)$$

Schwartz and Soffer [47] showed that the bulk exponents of the RF model obey the inequality $2\eta \geq \bar{\eta}$. The same arguments can be also applied to the surface correlation functions so that the surface exponents satisfy similar inequalities: $2\eta_\perp \geq \bar{\eta}_\perp$ and $2\eta_\parallel \geq \bar{\eta}_\parallel$. Note, that these inequality cannot be applied to the RA model where the coupling to disorder is bilinear.

III. FUNCTIONAL RENORMALIZATION GROUP

A. Perturbation theory

In the limit of low temperature and weak disorder the configuration of the system is fluctuating around the completely ordered state in which all replicas of all spins align along the same direction which is parallel to \mathbf{h} and \mathbf{h}_1 . It is convenient to split the order parameter $\mathbf{s}_a = (\sigma_a, \boldsymbol{\pi}_a)$ into the $(N-1)$ -component vector $\boldsymbol{\pi}_a$ which is perpendicular to this direction and the component $\sigma_a = \sqrt{1 - \boldsymbol{\pi}_a^2}$ parallel to this direction. Then the effective action of the system can be written as

$$\mathcal{S}[\boldsymbol{\pi}] = \frac{1}{T} \sum_{a=1}^n \left\{ \int_V \left[\frac{1}{2} (\nabla \boldsymbol{\pi}_a)^2 + \frac{(\boldsymbol{\pi}_a \cdot \nabla \boldsymbol{\pi}_a)^2}{2(1 - \boldsymbol{\pi}_a^2)} - h \sigma_a \right] - \int_S h_1 \sigma_a \right\} - \frac{1}{2T^2} \sum_{a,b=1}^n \int_V \mathcal{R}(\boldsymbol{\pi}_a \cdot \boldsymbol{\pi}_b + \sigma_a \sigma_b). \quad (17)$$

In general one has to add to the action (17) the terms like $\delta^d(0) \int_V \ln(1 - \boldsymbol{\pi}_a^2)$ generated by the Jacobian of the transformation from \mathbf{s}_a to $\boldsymbol{\pi}_a$. However, in what follows we will use the dimensional regularization scheme[48] in which $\delta^d(0) = 0$ so that we ignore these terms in action (17) from the beginning.

Let us denote averaging with the action (17) by double angular brackets and introduce the following correlation functions

$$G_{\alpha,\beta}^{(L,K)}(\mathbf{r}, \mathbf{x}) = \left\langle \left\langle \prod_{\nu=1}^L \pi_{\alpha_\nu}(\mathbf{r}_\nu) \prod_{\mu=1}^K \pi_{\beta_\mu}(\mathbf{x}_\mu) \right\rangle \right\rangle, \quad (18)$$

where L points $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_L)$ are off surface and K points $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K)$ are sitting on the surface. In Eq. (18) we have used a short notation $\alpha = (\alpha_1, \dots, \alpha_L)$ where each α_ν stands for the component number i_ν and the replica number a_ν . The similar holds for β . The correlation functions (18) can be computed using the following generating functional [49]

$$\mathcal{F}[\mathbf{J}, \mathbf{J}_1] = \ln \int \mathcal{D}\boldsymbol{\pi} e^{-\mathcal{S}[\boldsymbol{\pi}] + \int_V \mathbf{J}(\mathbf{r}) \boldsymbol{\pi}(\mathbf{r}) + \int_S \mathbf{J}_1(\mathbf{x}) \boldsymbol{\pi}(\mathbf{x})}, \quad (19)$$

where we assume that the source $\mathbf{J}(\mathbf{r})$ vanishes at the surface. Differentiating with respect to the sources we obtain

$$G_{\alpha,\beta}^{(L,K)}(\mathbf{r}, \mathbf{x}) = \left. \prod_{\nu=1}^L \frac{\delta}{\delta J(\mathbf{r}_\nu)} \prod_{\mu=1}^K \frac{\delta}{\delta J_1(\mathbf{x}_\mu)} \mathcal{F} \right|_{J=J_1=0}, \quad (20)$$

where for the sake of brevity we have suppressed all tensorial indices. Using correlation functions (18) one can compute the connected and disconnected functions defined in Eqs. (11) and (12). However, since we are interested only in the scaling behavior it is more convenient to consider the similar correlation functions not for \mathbf{s} but for $\boldsymbol{\pi}$ fields. For example, the correlation functions at two off surface points read

$$[\boldsymbol{\pi}(\mathbf{r}) \cdot \boldsymbol{\pi}(\mathbf{r}')]_{\text{con}} = \lim_{n \rightarrow 0} \sum_{i=1}^{N-1} G_{i,a;i,a}^{(2,0)}(\mathbf{r}, \mathbf{r}'), \quad (21)$$

$$[\boldsymbol{\pi}(\mathbf{r}) \cdot \boldsymbol{\pi}(\mathbf{r}')]_{\text{dis}} = \lim_{n \rightarrow 0} \sum_{i=1}^{N-1} G_{i,a;i,b}^{(2,0)}(\mathbf{r}, \mathbf{r}'). \quad (22)$$

where the connected correlator corresponds to a single replica and the disconnected one to two different replicas $a \neq b$. To compute the correlation functions at the surface like $[\boldsymbol{\pi}(\mathbf{r}) \cdot \boldsymbol{\pi}(\mathbf{x}')]_{\text{con}}$ or $[\boldsymbol{\pi}(\mathbf{x}) \cdot \boldsymbol{\pi}(\mathbf{x}')]_{\text{dis}}$ one has to replace $G^{(2,0)}$ by $G^{(1,1)}$ and $G^{(0,2)}$, respectively.

Expanding the effective action (17) in small $\boldsymbol{\pi}$ we will treat the quadratic part as a free action and the rest infinite series as interaction vertices (see Appendix A). Then the correlation functions (18) can be expressed in terms of Feynman diagrams which give the low temperature and small disorder expansion. In practical calculations it is convenient to perform the Fourier transform with respect to \mathbf{x} : $\hat{\boldsymbol{\pi}}(\mathbf{q}, z) = \int d^{d-1}x \boldsymbol{\pi}(\mathbf{x}, z) e^{-i\mathbf{q} \cdot \mathbf{x}}$ and define $\int_q := \int d^{d-1}q / (2\pi)^{d-1}$. The quadratic terms give the free propagator

$$\hat{G}_q^{(0)}(z, z') = \frac{1}{2\bar{q}} \left[e^{-\bar{q}|z-z'|} + \frac{\bar{q} + h_1}{\bar{q} - h_1} e^{-\bar{q}(z+z')} \right], \quad (23)$$

where we have introduced the short notation $\bar{q} := (q^2 + h)^{1/2}$. The free propagator (23) satisfies the boundary conditions

$$[\partial_z - h_1] G^{(0)}(\mathbf{x}, z, \mathbf{x}', z') \Big|_{z=0} = 0. \quad (24)$$

The free surface corresponds to the limit $h_1 \rightarrow 0$ in which Eq. (23) becomes the Neumann propagator consisting of the bulk part and the image part. In what follows we will use the Neumann propagator as the bare one and treat the terms proportional to h_1 as soft insertions [44, 50].

B. FRG equations and critical exponents

The correlation functions (18) calculated perturbatively in small disorder and temperature suffer of the UV divergences. To avoid mixture with IR singularities in the $O(N)$ -noninvariant correlation functions it is convenient to keep $\mathbf{h} \neq 0$. The UV divergences can be converted into poles in $\varepsilon = d - 4$ using dimensional regularization. To renormalize the theory one has to absorb these poles into finite number of Z -factors. However, all the Taylor coefficients r_μ of the disorder correlator $\mathcal{R}(\phi)$ turn out to be relevant operators so that one has to introduce renormalization of the whole function. To simplify calculation of the disorder renormalization one can use the background field method [25]. Using the Legendre transform of the generating functional (19) from the sources \mathbf{J} to the background fields $\boldsymbol{\Pi}$ one derives the effective action $\Gamma[\boldsymbol{\Pi}]$ which is the generating functional of the one-particle irreducible vertices. The two-replica part of the effective action gives the renormalization of the disorder. Since the scaling behavior is controlled by a zero temperature FP we will disregard all terms involving more than two replicas which are suppressed in the limit $T \rightarrow 0$. The renormalization of the disorder simplifies by changing variables: $\mathcal{R}(\phi) = R(z)$ where $z = \cos \phi$, for instance, $\mathcal{R}'(1) = -R''(0)$. In terms of the variable ϕ , the function $R(\phi)$ becomes periodic with the period 2π in the RF case and with the period π in the RA case. The relation between the renormalized and the bare correla-

tion functions reads

$$G^{(L,K)}(\mathbf{r}; T, h, h_1, R, \mu) = Z_\pi^{-(L+K)/2} Z_1^{-K/2} \times \hat{G}^{(L,K)}(\mathbf{r}; \hat{T}, \hat{h}, \hat{h}_1, \hat{R}). \quad (25)$$

where circles denote the bare quantities and μ is an arbitrary momentum scale. UV divergences are absorbed into Z -factors according to

$$\hat{\pi} = Z_\pi^{1/2} \pi, \quad \hat{\pi}|_s = (Z_\pi Z_1)^{1/2} \pi|_s, \quad (26)$$

$$\hat{h} = \mu^2 Z_T Z_\pi^{-1/2} h, \quad \hat{h}_1 = \mu Z_T (Z_\pi Z_1)^{-1/2} h_1, \quad (27)$$

$$\hat{T} = \mu^{2-d} Z_T T, \quad \hat{R} = \mu^{4-d} K_d^{-1} Z_R[R], \quad (28)$$

where $(2\pi)^d K_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a d -dimensional unit sphere and $\Gamma(x)$ is the Euler gamma function. $Z_R[R]$ in Eq. (28) is a functional acting on the renormalized disorder correlator $R(\phi)$ which has the following loop expansion:

$$Z_R[R] = R + \delta^{(1)}(R, R) + \delta^{(2)}(R, R, R) + \dots, \quad (29)$$

where $\delta^{(1)}(R, R)$ is bilinear in R and proportional to $1/\varepsilon$, while $\delta^{(2)}(R, R, R)$ is cubic in R and contains terms of order $1/\varepsilon$ and $1/\varepsilon^2$. According to Eq. (26) the surface field $\pi|_s$ renormalizes differently from the field π in the bulk. The new factor Z_1 serves to cancel the additional UV divergences in Feynman diagrams arising from the image part of the Neumann propagator $\hat{G}_q^{(0)}(z, z')$ for $z' \rightarrow 0$. The renormalized theory is not unique and depends on the scale μ . Using this fact we will derive the functional renormalization group (FRG) equation.

We now consider how the scaling behavior can be extracted from the renormalized theory. Using independence of the bare theory on the momentum scale μ one can derive the flow equations for the renormalized correlation functions differentiating the both sides of Eq. (25) with respect to μ at fixed bare quantities. One finds that the renormalized correlation functions satisfy the following FRG equation

$$\left[\mu \partial_\mu + (d-2-\zeta_T)T \partial_T - \zeta_h h \partial_h - \zeta_{h_1} h_1 \partial_{h_1} + \frac{L}{2} \zeta_\pi + \frac{K}{2} (\zeta_\pi + \zeta_1) - \int d\phi \beta[R(\phi)] \frac{\delta}{\delta R(\phi)} \right] G^{(L,K)} = 0, \quad (30)$$

where the integral in the last line is taken over a period, i.e., $(0, \pi)$ for RA and $(0, 2\pi)$ for RF models and we have introduced the scaling functions:

$$\zeta_i = \mu \partial_\mu \ln Z_i|_0, \quad (i = T, \pi, 1), \quad (31)$$

$$\zeta_h = 2 + \zeta_T - \zeta_\pi/2, \quad (32)$$

$$\zeta_{h_1} = 1 + \zeta_T - (\zeta_\pi + \zeta_1)/2, \quad (33)$$

$$\beta[R] = -\mu \partial_\mu R(\phi)|_0. \quad (34)$$

Here the zero indicates that the derivatives are taken at fixed bare quantities. Flow equations similar to Eq. (30) hold also for the correlation functions in which some or all the fields $\pi_a(\mathbf{r})$ are replaced by $\sigma_a(\mathbf{r})$ and for other observables, e.g., the correlation length and the magnetization [48].

The long-distance physics can be obtained from the solution of the FRG equation (30) in the limit of $\mu \rightarrow 0$. The renormalized disorder correlator and the temperature flow according to

$$-\mu \partial_\mu R(\phi) = \beta[R], \quad (35)$$

$$-\mu \partial_\mu \ln T = 2 - d + \zeta_T. \quad (36)$$

The scaling behavior is controlled by a zero temperature FP $\beta[R^*] = 0$ with R^* of order ε and $T^* = 0$. Indeed, according to Eq. (36), the temperature is irrelevant, i.e. it flows to 0 in the limit $\mu \rightarrow 0$ for $d > 2$ and for sufficiently small $\zeta_T = O(R)$. Although one expects that ζ_T is small in the vicinity of the FP, one has to take caution whether the zero temperature FP survives in three dimensions where $\zeta_T \sim \varepsilon$ is negative [26]. The stability of the FP can be checked by computing the eigenvalues of the disorder flow equation (35) linearized about the FP solution: $R(\phi) = R^*(\phi) + \sum_i t_i \Psi_i(\phi)$. Since one expects that for $d > 4$ ($\varepsilon > 0$) the FP $R^*(\phi)$ describes the paramagnetic-ferromagnetic transition it has to be unstable in a single direction $\Psi_1(\phi)$ with eigenvalue $\lambda_1 > 0$: $\beta[R^* + t_1 \Psi_1] = \lambda_1 t_1 \Psi_1 + O(t_1^2)$. In vicinity of the zero temperature FP which controls the paramagnetic-ferromagnetic transition, the FRG equation for the correlation length ξ can be written as

$$\left[\mu \partial_\mu - \lambda_1 t_1 \frac{\partial}{\partial t_1} \right] \xi(\mu, t_1) = 0. \quad (37)$$

Dimensional analysis implies that $\xi(\mu, t_1) = \mu^{-1} \bar{\xi}(t_1)$. This reduces Eq. (37) to an ordinary differential equation (ODE) which solution is $\xi \sim \mu^{-1} t_1^{-1/\lambda_1}$. The latter describes divergence of the correlation length on the critical line at zero temperature when the strength of disorder approaches the critical value [45]. Assuming that along the transition line at finite temperature $t_1 \sim T - T_c$ we find that the positive eigenvalue λ_1 gives the critical exponent of the correlation length $\nu = 1/\lambda_1$. For $d < 4$ ($\varepsilon < 0$) the FP becomes stable and describes a QLRO phase. The fluctuations exhibit power-law correlations in the whole QLRO phase so that the correlation length ξ is always infinite down to the lower critical dimension of the QLRO - paramagnetic transition.

Let us consider the solution of Eq. (30) for the connected two-point correlation functions. The dangerous irrelevance of the temperature manifests itself in the fact that the connected (bulk or surface) two point functions are proportional to T in the low temperature limit. This is explicitly shown in Appendix A for the connected correlation function $G^{(1,1)}$. Hence, setting $h = h_1 = 0$ and $R = R^*$ we can rewrite Eq. (30) as

$$\left[\mu \partial_\mu + \frac{1}{2} (L+K) \zeta_\pi^* + \frac{K}{2} \zeta_1^* + \theta \right] G_{\text{con}}^{(L,K)} = 0, \quad (38)$$

where the star denotes that the function is computed at the FP. In Eq. (38) we have defined the exponent

$$\theta = d - 2 - \zeta_T^*, \quad (39)$$

which describes the flow of the temperature (36) in the vicinity of the FP and which has been introduced ad hoc in the

modified hyperscaling relation (7). Using the method of characteristics and dimensional analysis one can write the solution of Eq. (38) in the form

$$G_{\text{con}}^{(L,K)}(rb; R^*) = b^{-(\frac{1}{2}(L+K)\zeta_\pi^* + K\zeta_1^*/2 + \theta)} f_c(r; R^*). \quad (40)$$

Considering the connected two point functions (40) with $(L = 2, K = 0)$, $(L = 1, K = 1)$, and $(L = 0, K = 2)$ we derive the critical exponents:

$$\eta = \zeta_\pi^* - \zeta_T^*, \quad (41)$$

$$\eta_\perp = \zeta_\pi^* + \zeta_1^*/2 - \zeta_T^*, \quad (42)$$

$$\eta_\parallel = \zeta_\pi^* + \zeta_1^* - \zeta_T^*. \quad (43)$$

We next turn to the disconnected two-point correlation functions. At variance with the connected correlation functions they are not proportional to the temperature. Thus, at $h = h_1 = T = 0$ they satisfy the same Eq. (38) but without the term θ in the brackets. The solution of the latter FRG equation is given by

$$G_{\text{dis}}^{(L,K)}(rb; R^*) = b^{-(\frac{1}{2}(L+K)\zeta_\pi^* + K\zeta_1^*/2)} f_d(r; R^*). \quad (44)$$

Repeating analysis we did for the connected functions we arrive at

$$\bar{\eta} = 4 - d + \zeta_\pi^* = 2 + \eta - \theta, \quad (45)$$

$$\bar{\eta}_\perp = 4 - d + \zeta_\pi^* + \zeta_1^*/2 = 2 + \eta_\perp - \theta, \quad (46)$$

$$\bar{\eta}_\parallel = 4 - d + \zeta_\pi^* + \zeta_1^* = 2 + \eta_\parallel - \theta. \quad (47)$$

Note that the exponents (41)-(43) and (45)-(47) are related by

$$2\eta_\perp = \eta + \eta_\parallel, \quad 2\bar{\eta}_\perp = \bar{\eta} + \bar{\eta}_\parallel. \quad (48)$$

Finally we study the profile of the spontaneous magnetization below and at the paramagnetic-ferromagnetic transition for $d > d_{lc}$. The magnetization as a function of the distance to the surface z , the reduced temperature t_1 , and the bulk and surface magnetic fields h and h_1 satisfies the following flow equation

$$\left[\mu \partial_\mu - \zeta_h^* h \partial_h - \zeta_{h_1}^* h_1 \partial_{h_1} + \frac{1}{2} \zeta_\pi^* + \frac{j}{2} \zeta_1^* - \lambda_1 t_1 \frac{\partial}{\partial t_1} \right] \sigma(z, t_1, h, h_1) = 0. \quad (49)$$

Here $j = 0$ and $z > 0$ corresponds to the bulk magnetization σ while $j = 1$ and $z = 0$ gives the surface magnetization σ_1 . The solution of Eq. (49) can be written as

$$\sigma(z, t_1, h, h_1) = b^{-(\frac{1}{2}\zeta_\pi^* + \frac{j}{2}\zeta_1^*)} \times \sigma(zb^{-1}, t_1b^{\lambda_1}, hb^{\zeta_h^*}, h_1b^{\zeta_{h_1}^*}). \quad (50)$$

We first consider the profile for $h = h_1 = 0$. The solution (50) interpolates between the surface magnetization $\sigma_1(t_1) \sim t_1^{(\zeta_\pi^* + \zeta_1^*)/(2\lambda_1)}$ at $z \approx 0$ and the bulk magnetization $\sigma(t_1, z) \sim t_1^{\zeta_\pi^*/(2\lambda_1)}$ for $z \gg \xi$. Reexpressing the latter in terms of $\nu, \bar{\eta}$, and $\bar{\eta}_\parallel$ we obtain that the bulk and the surface magnetization exponents defined in Eq. (8) are given by

$$\beta = \frac{1}{2}\nu(d - 4 + \bar{\eta}), \quad \beta_1 = \frac{1}{2}\nu(d - 4 + \bar{\eta}_\parallel). \quad (51)$$

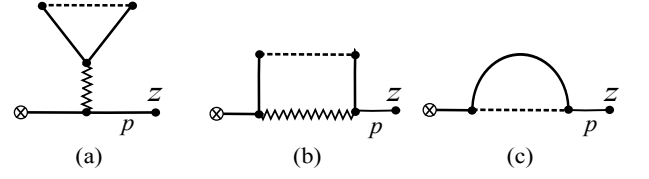


FIG. 1. One-loop diagrams contributing to the connected two-point function $\hat{G}_{1,a;1,a}^{(1,1)}(z, p)$. The solid lines stand for the Neumann propagator (23). The wavy and dashed lines are vertices defined in Eqs. (A1)-(A3). The crossed circles denote the points on the surface.

At the critical point $t_1 = 0$ and finite external fields we find that $\sigma(h) \sim h^{\zeta_\pi^*/(2\zeta_h^*)}$ in the bulk and $\sigma_1(h) \sim h^{(\zeta_\pi^* + \zeta_1^*)/(2\zeta_h^*)}$ or $\sigma_1(h_1) \sim h_1^{(\zeta_\pi^* + \zeta_1^*)/(2\zeta_{h_1}^*)}$ at the surface. Thus, the exponents δ , δ_1 , and δ_{11} defined in Eqs. (9) and (10) satisfy the following scaling relations:

$$\frac{\delta - 1}{2 - \eta} = \frac{\nu}{\beta}, \quad \frac{\delta_1 - \beta/\beta_1}{2 - \eta} = \frac{\nu}{\beta_1}, \quad \frac{\delta_{11} - 1}{1 - \eta_\parallel} = \frac{\nu}{\beta_1}. \quad (52)$$

IV. THE SURFACE EXPONENTS TO ONE-LOOP ORDER

We now renormalize the both semi-infinite RF and RA models to one-loop order and explicitly calculate the surface critical exponents to first order in $\varepsilon = d - 4$. The factors Z_π , Z_T and $Z_R[R]$ defined in Eqs. (26)-(29) are the same that appear in the case of the infinite systems. They have been calculated in several works up to two-loop order [21, 26–29]. To one-loop order they read

$$Z_\pi = 1 - (N - 1) \frac{R''(0)}{\varepsilon} + O(R^2), \quad (53)$$

$$Z_T = 1 - (N - 2) \frac{R''(0)}{\varepsilon} + O(R^2), \quad (54)$$

$$\varepsilon \delta^{(1)}(R, R) = \frac{1}{2} R''(\phi)^2 - R''(0) R''(\phi) - (N - 2) \left\{ R''(0) [2R(\phi) + R'(\phi) \cot \phi] - \frac{1}{2 \sin^2 \phi} [R'(\phi)]^2 \right\}. \quad (55)$$

The new factor Z_1 which eliminates the poles resulting from the presence of the surface can be determined from the renormalization of the two point function $\hat{G}^{(1,1)}(p, z; \hat{h}, \hat{T}, \hat{R})$. The one-loop diagrams contributing to this function are shown in Fig. 1. The corresponding integrals are computed in Appendix A and give

$$\hat{G}^{(1,1)}(p, z; \hat{h}, \hat{T}, \hat{R}) = \hat{T} \frac{e^{-\bar{p}z}}{\bar{p}} \left\{ 1 - \frac{K_d}{4\varepsilon} \hat{R}''(0) \times \left[(N - 3) \left(\frac{\hat{h}}{\bar{p}^2} + \frac{z\hat{h}}{\bar{p}} \right) + 2(N + 1) \right] + O(\hat{R}^2) \right\}, \quad (56)$$

where $\bar{p} = (p^2 + \hat{h}^2)^{1/2}$. The factor Z_1 can be found from the renormalization condition

$$Z_\pi^{-1} Z_1^{-1/2} \hat{G}^{(1,1)}(p, z; \hat{h}, \hat{T}, \hat{R}) = \text{finite for } \varepsilon \rightarrow 0, \quad (57)$$

where the bare $\hbar, \tilde{T}, \tilde{R}$ are replaced by the renormalized h, T and R according to Eqs. (26)-(28). We obtain

$$Z_1 = 1 - (N-1) \frac{R''(0)}{\varepsilon} + O(R^2). \quad (58)$$

Thus, to one loop order we have $Z_1 = Z_\pi + O(R^2)$. Using Eqs. (31) and (34) we calculate the scaling functions

$$\zeta_T = -(N-2)R''(0) + O(R^2), \quad (59)$$

$$\zeta_\pi = \zeta_1 = -(N-1)R''(0) + O(R^2), \quad (60)$$

and the beta function

$$\begin{aligned} \beta[R] = & -\varepsilon R(\phi) + \frac{1}{2}R''(\phi)^2 - R''(0)R''(\phi) \\ & - (N-2) \left\{ R''(0)[2R(\phi) + R'(\phi) \cot \phi] \right. \\ & \left. - \frac{1}{2\sin^2 \phi} [R'(\phi)]^2 \right\} + O(R^2) \end{aligned} \quad (61)$$

to one-loop order. Solution of the FP equation $\beta[R^*] = 0$ with the beta function (61) has been analyzed for different values of N and different sign of ε in Refs. [26–29]. We first assume for granted that the flow has a FP $R^*(\phi)$ which is a π -periodic function for the RA model and a 2π -periodic function for the RF model. Then, the surface critical exponents can be computed to one loop using Eqs. (41)-(43) and (45)-(47) that yields

$$\eta = -R^{*''}(0), \quad \bar{\eta} = -\varepsilon - (N-1)R^{*''}(0), \quad (62)$$

$$\eta_\perp = -\frac{N+1}{2}R^{*''}(0), \quad \bar{\eta}_\perp = -\varepsilon - \frac{3}{2}(N-1)R^{*''}(0), \quad (63)$$

$$\eta_\parallel = -NR^{*''}(0), \quad \bar{\eta}_\parallel = -\varepsilon - 2(N-1)R^{*''}(0). \quad (64)$$

The other surface exponents are related to (62)-(64) by the scaling relations (51) and (52).

Before we explicitly calculate the surface exponents for the semi-infinite RF and RA models let us remind how the FRG allows one to overcome the DR problem. The incorrect DR prediction results from the assumption that the flow equation (35) with the beta function (61) has a FP which is an analytic function. Indeed, in this case one can obtain a closed flow equation for the $R''(0)$:

$$-\mu \partial_\mu R''(0) = -\varepsilon R''(0) - (N-2)R''(0)^2. \quad (65)$$

Equation (65) has a nontrivial FP solution $R^{*''}(0) = -\varepsilon/(N-2)$ with the eigenvalue $\lambda_1 = \varepsilon$. This FP is unstable for $\varepsilon > 0$ as one expects for a FP corresponding to the transition and gives the DR exponents: $\nu^{(\text{DR})} = 1/\varepsilon$ and

$$\eta^{(\text{DR})} = \bar{\eta}^{(\text{DR})} = \frac{\varepsilon}{N-2}, \quad (66)$$

$$\eta_\perp^{(\text{DR})} = \bar{\eta}_\perp^{(\text{DR})} = \frac{N+1}{2(N-2)}\varepsilon, \quad (67)$$

$$\eta_\parallel^{(\text{DR})} = \bar{\eta}_\parallel^{(\text{DR})} = \frac{N}{N-2}\varepsilon. \quad (68)$$

$$(69)$$

The one-loop DR exponents for the magnetization read

$$\beta^{(\text{DR})} = \frac{N-1}{2(N-2)}, \quad \beta_1^{(\text{DR})} = \frac{N-1}{N-2}. \quad (70)$$

For $\varepsilon < 0$ the FP is stable but the η critical exponents become negative, and hence, unphysical.

More accurate analysis of the RG flow shows that $R'''(0)$ diverges at a finite scale μ . Thus, no analytic FP can exist and one has to look for a non-analytic FP with $R^{*'''}(0^+) \neq 0$ which would violate the DR predictions. This requires solution of the boundary-value problem for the nonlinear ODE with periodic boundary conditions, which depend on the universality class. We assume that the small ϕ expansion of the FP solution $R^*(\phi)$ has the following form

$$R^*(\phi) = a_0 + a_2\phi^2 + a_3|\phi|^3 + a_4\phi^4 + a_5|\phi|^5 + \dots, \quad (71)$$

meaning that $R^{*''}(\phi)$ has a cusp at the origin with $R^{*'''}(0^+) \neq 0$. Substituting ansatz (71) into the FP equation we find that the first coefficients are given by

$$a_0 = -\frac{2a_2^2(N-1)}{4(N-2)a_2 + \varepsilon}, \quad a_2 = \frac{R^{*''}(0)}{2}, \quad (72)$$

$$a_3 = -\text{sign}(\varepsilon) \sqrt{\frac{2\varepsilon a_2 + 4a_2^2(N-2)}{9(N+2)}}. \quad (73)$$

The value of $R^{*''}(0)$ as well as the sign of a_3 are constrained by the boundary conditions. $R^{*''}(0)$ can be determined using the shooting method to fulfill the appropriate periodicity requirement.

A. Random field $O(N)$ model

1. Paramagnetic-ferromagnetic transition for $d > 4$ ($\varepsilon > 0$)

The RF model is described by $R(\phi)$ which is a 2π -periodic function. Numerical solution of the FP equation shows that for $d > 4$ a 2π -periodic solution of the form (71)-(73) exists only for $N > N_c = 2.83474$. It has $R^{*''}(0) < 0$ and it disappears when $N \rightarrow N_c^+$. This cuspy FP is once unstable with the positive eigenvalue $\lambda_1 = \varepsilon$. Thus, the correlation length exponent $\nu = 1/\varepsilon + (\varepsilon^0)$ coincides with the DR prediction to one-loop order. Remarkably, the non-zero $R^{*'''}(0^+)$ vanishes for $N > N^* = 18 + O(\varepsilon)$. The non-analyticity becomes weaker as N increases and starts with $R^{*(2p(N)+1)}(0^+) \neq 0$ where $p \sim N$ [28, 29, 52]. Weaker non-analyticity results in restoring the DR critical exponents for $N > N^*$. The critical exponents η_i and $\bar{\eta}_i$ computed using Eqs. (62)-(64) as functions of N are shown in Fig. 2. With increasing N they monotonically decay approaching the DR values at $N = N^*$ and satisfying the inequalities: $\eta < \bar{\eta} < \eta_\perp < \bar{\eta}_\perp < \eta_\parallel < \bar{\eta}_\parallel$. The bulk and surface magnetization exponent β and β_1 calculated for different N are shown in inset of Fig. 2. To one-loop order they obey relation $\beta_1 = 2\beta$. Up to now the both magnetization exponents have been studied only for the 3D RFIM where numerical simulations give $\beta = 0.0017 \pm 0.005$ [51] and $\beta_1 = 0.23 \pm 0.03$ [42]. Thus, the ratio β_1/β for the RF $O(N)$ systems in $d > 4$ is much smaller than for the 3D RFIM.

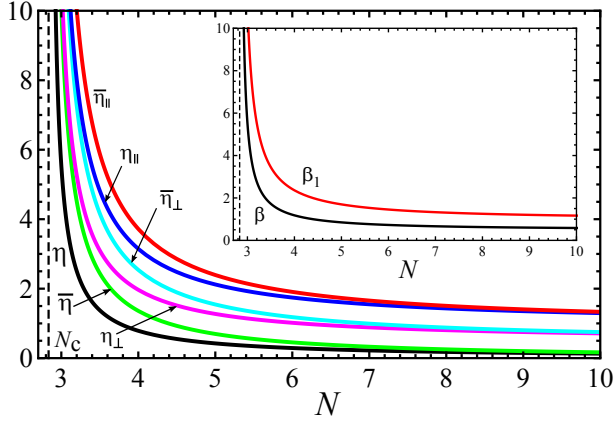


FIG. 2. (Color online) The critical exponents η_i and $\bar{\eta}_i$ (divided by ε), which describe the paramagnetic-ferromagnetic transition of the RF model above the lower critical dimension, as functions of N for $N > N_c$. Inset: The corresponding bulk magnetization exponent β and the surface magnetization exponent β_1 as functions of N .

2. Quasi-long-range order for $d < 4$ ($\varepsilon < 0$)

Below the lower critical dimension the flow equation for the disorder correlator has an attractive 2π -periodic FP solution of the form (71)-(73). This cuspy FP appears only for $2 \leq N < N_c$ where it controls the scaling behavior of spin fluctuations in the QLRO phase. The corresponding exponents η_i and $\bar{\eta}_i$ as functions of N are shown in Fig. 3. In the case $N = 2$ the FP equation admits for an explicit non-analytic ϕ_0 -periodic solution given by

$$R^*(\phi) = \frac{|\varepsilon|\phi_0^4}{72} \left[\frac{1}{36} - \left(\frac{\phi}{\phi_0} \right)^2 \left(1 - \frac{\phi}{\phi_0} \right)^2 \right]. \quad (74)$$

Using Eqs. (62)-(64) one obtains

$$\eta = \frac{\phi_0^2}{36} |\varepsilon|, \quad \bar{\eta} = \left(1 + \frac{\phi_0^2}{36} \right) |\varepsilon|, \quad (75)$$

$$\eta_{\perp} = \frac{\phi_0^2}{24} |\varepsilon|, \quad \bar{\eta}_{\perp} = \left(1 + \frac{\phi_0^2}{24} \right) |\varepsilon|, \quad (76)$$

$$\eta_{\parallel} = \frac{\phi_0^2}{18} |\varepsilon|, \quad \bar{\eta}_{\parallel} = \left(1 + \frac{\phi_0^2}{18} \right) |\varepsilon|, \quad (77)$$

with $\phi_0 = 2\pi$ for the RF system.

The semi-infinite RF $O(2)$ model can be mapped onto a semi-infinite periodic disordered elastic system with a free surface. There is one to one correspondence between the Bragg glass phase of the elastic system and the QLRO phase of the studied spin model. The power-law decay of the spin correlations in the QLRO phase corresponds to the logarithmic growth of the displacements in the disordered elastic system. Moreover, the exponents η , η_{\perp} and η_{\parallel} provide the universal amplitudes of the logarithmic growth of the displacements in the bulk, at the surface and along the surface, respectively. For a ϕ_0 -periodic elastic system with a free surface these amplitudes are given by Eqs. (75)-(75). In particular, we find that the logarithmic growth of the displacements along the

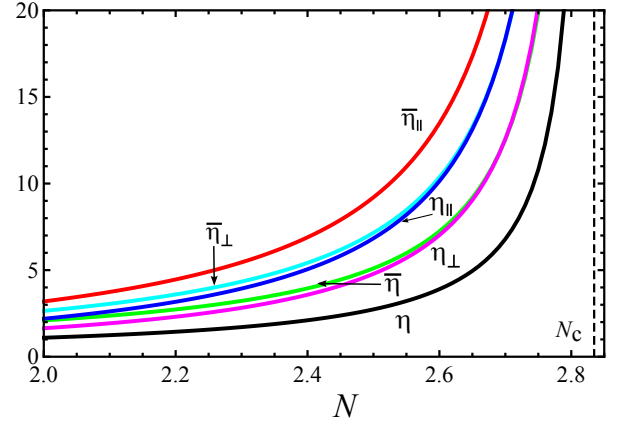


FIG. 3. (Color online) The critical exponents η_i and $\bar{\eta}_i$ (divided by $|\varepsilon|$), which describe the power-law decay of correlations in the QLRO phase of the RF model below the lower critical dimension, as functions of N for $N < N_c$.

surface is twice larger than the logarithmic growth in the bulk. In the case when only one point is on the surface the growth is enhanced by 50%. The presence of a free surface can be considered as an extended defect of a special kind. The influence of potential-like extended defects on the Bragg-glass has been recently studied in Refs. [30, 54].

B. Random anisotropy $O(N)$ model

1. Paramagnetic-ferromagnetic transition for $d > 4$ ($\varepsilon > 0$)

The FP equation has a cuspy π -periodic solution of the form (71)-(73) which is singly unstable giving the correlation length exponent $\nu = 1/\varepsilon + (\varepsilon^0)$. It exists for any $N > N_c = 9.4412$ with a non-zero $R^{*'''}(0^+)$. Therefore, at variance with the RF case in the RA model the DR breaks down for all values $N > N_c$, i.e., $N^* = \infty$ [29]. The N -dependence of the critical exponents η_i and $\bar{\eta}_i$ is shown in Fig. 4. For large N one can find the asymptotic behavior of the FP solution [28, 29, 52, 53]. Following Ref. [28] we look for the π -periodic solution of the FP equation $\beta[R] = 0$ with the beta function (61) of the form

$$R^{*'}(\phi) = -\frac{3}{2}\delta\varepsilon \sin\left(\frac{\pi-2\phi}{3}\right) (2x(\phi) - 1) G(x). \quad (78)$$

Here we have introduced a small parameter $\delta = 1/(N-2)$ and defined variable $x(\phi) = \cos(\frac{\pi-2\phi}{3})$. Substituting ansatz (78) into the FP equation and expanding the function $G(x)$ in small δ one finds that the coefficients are polynomials in x :

$$\begin{aligned} G(x) = 1 + \frac{2}{9}(95 - 44x - 16x^2)\delta - \frac{4}{81}(11737 - 5040x \\ - 3624x^2 - 3104x^3 - 768x^4)\delta^2 + \frac{8}{10935}(103378933 \\ - 45854072x - 23128624x^2 - 16172328x^3 \\ - 9791216x^4 - 4642048x^5 - 901120x^6)\delta^3 + O(\delta^4). \end{aligned} \quad (79)$$

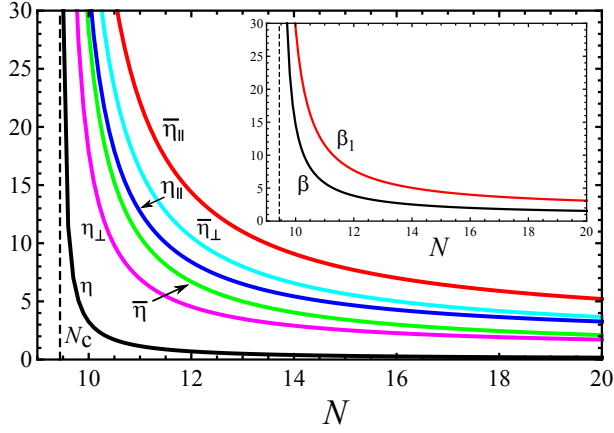


FIG. 4. (Color online) The critical exponents η_i and $\bar{\eta}_i$ (divided by ε), which describe the paramagnetic-ferromagnetic transition of the RA model above the lower critical dimension, as functions of N for $N > N_c$. Inset: The corresponding bulk magnetization exponent β and the surface magnetization exponent β_1 as functions of N .

This implies that [53]

$$\frac{R^{*''}(0)}{\varepsilon\delta} = -\frac{3}{2} - 23\delta + \frac{1750}{3}\delta^2 - \frac{2129692}{27}\delta^3 + O(\delta^4). \quad (80)$$

Substituting the solution (80) into Eqs. (62)-(64) we find the correlation function exponents to leading order in $1/N$ as:

$$\eta = \frac{3\varepsilon}{2N} \left(1 + \frac{52}{3N} + \dots \right), \quad \bar{\eta} = \frac{\varepsilon}{2} \left(1 + \frac{49}{N} + \dots \right), \quad (81)$$

$$\eta_{\perp} = \frac{3\varepsilon}{4} \left(1 + \frac{55}{3N} + \dots \right), \quad \bar{\eta}_{\perp} = \frac{5\varepsilon}{4} \left(1 + \frac{147}{5N} + \dots \right), \quad (82)$$

$$\eta_{\parallel} = \frac{3\varepsilon}{2} \left(1 + \frac{52}{3N} + \dots \right), \quad \bar{\eta}_{\parallel} = 2\varepsilon \left(1 + \frac{49}{2N} + \dots \right), \quad (83)$$

$$\beta = \frac{3}{4} \left(1 + \frac{49}{3N} + \dots \right), \quad \beta_1 = \frac{3}{2} \left(1 + \frac{49}{3N} + \dots \right), \quad (84)$$

where in the last line are the bulk and the surface magnetization exponents.

2. Quasi-long-range order for $d < 4$ ($\varepsilon < 0$)

For $2 \leq N < N_c$ the flow equation has a stable π -periodic FP solution of the form (71)-(73) which controls the scaling behavior of spin fluctuations in the QLRO phase of the RA model for $d < 4$. The correlation function exponents η_i and $\bar{\eta}_i$ computed for different N are shown in Fig. 5. For $N = 2$ the FP equation has an explicit non-analytic π -periodic solution given by Eq. (74) with $\phi_0 = \pi$. The critical exponents of the RA $O(2)$ model are given by Eqs. (75)-(77) with $\phi_0 = \pi$.

V. SUMMARY

In the present work, we have investigated the RF and RA semi-infinite $O(N)$ models with a free surface. The both

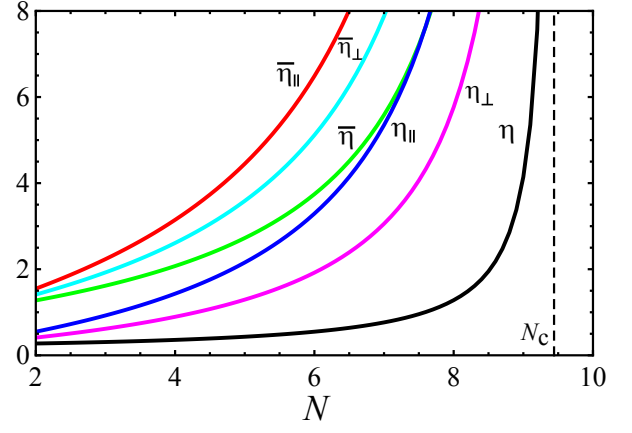


FIG. 5. (Color online) The critical exponents η_i and $\bar{\eta}_i$ (divided by $|\varepsilon|$), which describe the power-law decay of correlations in the QLRO phase of the RA model below the lower critical dimension, as functions of N for $N < N_c$.

models have the lower critical dimension $d_{lc} = 4$. Above d_{lc} they undergo a paramagnetic-ferromagnetic transition for $N > N_c$, while below d_{lc} and for $N < N_c$ they exhibit a QLRO phase with zero magnetization and power-law correlation of spins. Here the critical value $N_c = 2.835$ for the RF models and $N_c = 9.441$ for the RA. Using FRG we studied the surface scaling behavior of these models at criticality as well as in the QLRO phase, and calculate the corresponding surface exponents to lowest order in $\varepsilon = d - 4$. We have found that the DR prediction for the surface scaling is broken similar to that happens in the bulk. We have shown that the connected and disconnected correlation functions scale differently also at the surface and derived the scaling relations between different surface exponents. The surface exponents obtained for the 3D RF $O(2)$ can be used to describe the growth of displacements near a free surface in semi-infinite periodic elastic systems in disordered media. The surface scaling we obtained for the Heisenberg ($N = 3$) RA model can be relevant for the behavior of amorphous magnets [10, 55].

ACKNOWLEDGMENTS

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Appendix A: One-loop diagrams contributing to $\hat{G}^{(1,1)}$

In this appendix we calculate the correlation function $\hat{G}_{1,a;1,a}^{(1,1)}$ to one-loop order. Expanding action (17) in small

π_a we find that the only vertices we need are

$$\begin{array}{c} ia \\ \diagdown \\ z_1 \\ \diagup \\ ia \end{array} \begin{array}{c} q \\ \text{---} \\ z_2 \\ \text{---} \\ ja \end{array} \begin{array}{c} ja \\ \diagup \\ z_2 \\ \diagdown \\ ja \end{array} = -\frac{1}{8\dot{T}} \delta(z_1 - z_2) \left[q^2 + \partial_{z_1} \partial_{z_2} + \dot{h} \right], \quad (A1)$$

$$\begin{array}{c} ia \\ \diagdown \\ z \\ \diagup \\ ia \end{array} \begin{array}{c} q \\ \text{---} \\ z \\ \text{---} \\ jb \end{array} = -\frac{1}{2\dot{T}^2} \dot{R}''(0), \quad (A2)$$

$$\begin{array}{c} ia \\ \diagdown \\ z \\ \diagup \\ ia \end{array} \begin{array}{c} q \\ \text{---} \\ z \\ \text{---} \\ jb \end{array} = -\frac{1}{8\dot{T}^2} \dot{R}''(0). \quad (A3)$$

The one loop diagrams contributing to the correlation function $\dot{G}_{1,a;1,a}^{(1,1)}$ are shown in Fig. 1. The solid line corresponds to the Neumann propagator (23) with $\mathbf{h}_1 = 0$ and the wavy lines to vertices (A1)-(A3). The first diagram gives

$$\begin{aligned} (a) &= \frac{N-1}{2\dot{T}^3} \dot{R}''(0) \int_q \int_0^\infty dz_1 \int_0^\infty dz_2 \int_0^\infty dz_3 \delta(z_2 - z_1) \\ &\times \left[\partial_{z_1} \partial_{z_2} + \dot{h} \right] G_p^{(0)}(0, z_1) G_p^{(0)}(z_1, z) \left[G_q^{(0)}(z_2, z_3) \right]^2 \\ &= \frac{N-1}{16\bar{p}} \dot{T} \dot{R}''(0) e^{-\bar{p}z} \left(\frac{z\dot{h}}{\bar{p}} + \frac{\dot{h}}{\bar{p}^2} + 2 \right) I_2 + \text{finite}, \quad (A4) \end{aligned}$$

where we have used $\bar{p} = (p^2 + \dot{h}^2)^{1/2}$ and omitted the terms finite in the limit $\varepsilon \rightarrow 0$. The logarithmically divergent one-

loop integral reads

$$\begin{aligned} I_2 &= \int_q \frac{1}{\bar{q}(\bar{p} + \bar{q})^2} = K_{d-1} \int_0^\infty \frac{q^{d-2} dq}{(q^2 + \dot{h}^2)^{3/2}} + O(\varepsilon^0) \\ &= -\frac{4K_d}{\varepsilon} + O(\varepsilon^0). \quad (A5) \end{aligned}$$

The second and third diagrams yield

$$\begin{aligned} (b) &= \frac{1}{\dot{T}^3} \dot{R}''(0) \int_q \int_0^\infty dz_1 \int_0^\infty dz_2 \int_0^\infty dz_3 \delta(z_2 - z_1) \\ &\times \left[(\mathbf{p} + \mathbf{q})^2 + \partial_{z_1} \partial_{z_2} + \dot{h} \right] G_p^{(0)}(0, z_1) G_p^{(0)}(z_2, z) \\ &\times G_q^{(0)}(z_1, z_3) G_q^{(0)}(z_3, z_2) = -\frac{1}{8\bar{p}} \dot{T} \dot{R}''(0) e^{-\bar{p}z} \\ &\times \left[\left(\frac{z\dot{h}}{\bar{p}} + \frac{\dot{h}}{\bar{p}^2} - 2\bar{p}z - 7 \right) I_2 - 2I_3 \right] + \text{finite}, \quad (A6) \end{aligned}$$

$$\begin{aligned} (c) &= -\frac{1}{\dot{T}^2} \dot{R}''(0) \int_q \int_0^\infty dz_1 G_p^{(0)}(0, z_1) G_q^{(0)}(z_1, z_1) \\ &\times G_p^{(0)}(z_1, z) = -\frac{1}{8\bar{p}} \dot{T} \dot{R}''(0) e^{-\bar{p}z} \\ &\times [(2\bar{p}z + 5) I_2 + 2I_3] + \text{finite}, \quad (A7) \end{aligned}$$

where we have defined the algebraically divergent integral

$$I_3(\bar{p}, z) = \int_q \frac{3\bar{p} + \bar{q} + (2\bar{p} + \bar{q})\bar{p}z}{\bar{p}^2(\bar{p} + \bar{q})^2}. \quad (A8)$$

Summing up the three diagrams we find that the algebraically divergent integral (A8) cancels and we obtain Eq. (56).

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